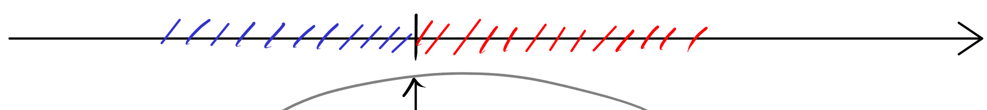
Returning to the example above:

The constraints of D ensure that the value of any sol. to D is a lower bound on the value of any sol. to P, i.e., for any pair x, y of sol. to P and D resp.,

$$10y_1 + 6y_2 \leq 7x_1 + x_2 + 5x_3$$ Weak duality



opt. value for both problems — Strong duality

Consider again the inequality leading to the constraints of the dual:

$$7x_1 + x_2 + 5x_3 \geq y_1(x_1 - x_2 + 3x_3) + y_2(5x_1 + 2x_2 - x_3)$$

Looking at the righthand side:

$y_1 = 0 \lor x_1 - x_2 + 3x_3 = 10 \iff$ $\qquad$ $y_2 = 0 \lor 5x_1 + 2x_2 - x_3 = 6 \iff$

$$\underbrace{y_1(x_1 - x_2 + 3x_3)}_{=10y_1} + \underbrace{y_2(5x_1 + 2x_2 - x_3)}_{=6y_2}$$

$$= \underbrace{(y_1 + 5y_2)x_1}_{=7x_1} + \underbrace{(-y_1 + 2y_2)x_2}_{=x_2} + \underbrace{(3y_1 - y_2)x_3}_{=5x_3}$$

$\Updownarrow$ $y_1 + 5y_2 = 7 \lor x_1 = 0$ $\qquad$ $\Updownarrow$ $-y_1 + 2y = 1 \lor x_2 = 0$ $\qquad$ $\Updownarrow$ $3y_1 - y_2 = 5 \lor x_3 = 0$

Thus,

$$7x_1 + x_2 + 5x_3 = 10y_1 + 6y_2$$

$$\Updownarrow$$

Complementary Slackness Conditions (c.s.c.)

$$\left\{\begin{array}{llll} x_1 = 0 & \vee & y_1 + 5y_2 = 7 & \\ x_2 = 0 & \vee & -y_1 + 2y_2 = 1 & \text{primal c.s.c.} \\ x_3 = 0 & \vee & 3y_1 - y_2 = 5 & \\ y_1 = 0 & \vee & x_1 - x_2 + 3x_3 = 10 & \\ y_2 = 0 & \vee & 5x_1 + 2x_2 - x_3 = 6 & \text{dual c.s.c.} \end{array}\right.$$

Since $p \vee q \equiv \neg p \Rightarrow q$, this can also be written as:

$$7x_1 + x_2 + 5x_3 = 10y_1 + 6y_2$$

$$\Updownarrow$$

Complementary Slackness Conditions (c.s.c.)

$$\left\{\begin{array}{llll} x_1 > 0 & \Rightarrow & y_1 + 5y_2 = 7 & \\ x_2 > 0 & \Rightarrow & -y_1 + 2y_2 = 1 & \text{primal c.s.c.} \\ x_3 > 0 & \Rightarrow & 3y_1 - y_2 = 5 & \\ y_1 > 0 & \Rightarrow & x_1 - x_2 + 3x_3 = 10 & \\ y_2 > 0 & \Rightarrow & 5x_1 + 2x_2 - x_3 = 6 & \text{dual c.s.c.} \end{array}\right.$$

By The Strong Duality Theorem (which we will not prove), there exist solutions fulfilling the c.s.c.

Moreover, if the c.s.c. are „close" to being satisfied, the values of the primal and dual sol. are „close":

Relaxed Complementary Slackness Conditions

$$\begin{cases} x_1 > 0 & \Rightarrow & y_1 + 5y_2 \geq 7/b \\ x_2 > 0 & \Rightarrow & -y_1 + 2y_2 \geq 1/b \\ x_3 > 0 & \Rightarrow & 3y_1 - y_2 \geq 5/b \\ & & \\ y_1 > 0 & \Rightarrow & x_1 - x_2 + 3x_3 \leq 10c \\ y_2 > 0 & \Rightarrow & 5x_1 + 2x_2 - x_3 \leq 6c \end{cases}$$

$$\Downarrow$$

$$7x_1 + x_2 + 5x_3 \leq bc(10y_1 + 6y_2)$$

Proof:

$$(y_1 + 5y_2)x_1 + (-y_1 + 2y_2)x_2 + (3y_1 - y_2)x_3$$

$$\geq \frac{7}{b}x_1 + \frac{1}{b}x_2 + \frac{5}{b}x_3, \quad \text{by the Primal relaxed c.s.c.}$$

$$= \frac{1}{b}(7x_1 + x_2 + 5x_3)$$

$$\Downarrow$$

$$7x_1 + x_2 + 5x_3 \leq b\big((y_1 + 5y_2)x_1 + (-y_1 + 2y_2)x_2 + (3y_1 - y_2)x_3\big)$$

$$= b\big((x_1 - x_2 + 3x_3)y_1 + (5x_1 + 2x_2 - x_3)y_2\big)$$

$$\leq b(10cy_1 + 6cy_2), \quad \text{by the Dual r.c.s.c.}$$

$$= bc(10y_1 + 6y_2)$$