Section 1.7: Randomized Rounding

Alg RR_1

Solve LP
$I \leftarrow \emptyset$
For $j \leftarrow 1$ to m
    With probability x_j
        $I \leftarrow I \cup \{j\}$

$E[w(I)] = Z^*_{LP} \leq OPT$,
but the result is most likely not a set cover.

Alg RR_2

Solve LP
  $I \leftarrow \emptyset$
  For $i \leftarrow 1$ to $2 \cdot \ln(n)$
    For $j \leftarrow 1$ to m
      With probability x_j
        $I \leftarrow I \cup \{j\}$

$E[w(I)] \leq 2 \cdot \ln(n) \cdot Z^*_{LP} \leq 2 \cdot \ln(n) \cdot OPT$,
and with high prob. all elements are covered.
(Calculations below)

Alg RR_3

Solve LP

Repeat

$\quad I \leftarrow \emptyset$

$\quad$ For $i \leftarrow 1$ to $2 \cdot \ln(n)$

$\quad\quad$ For $j \leftarrow 1$ to m

$\quad\quad\quad$ With probability x_j

$\quad\quad\quad\quad I \leftarrow I \cup \{j\}$

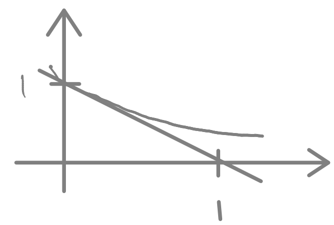
Until $\{S_j \mid j \in I\}$ is a set cover and $w(I) \leq 4 \ln(n) Z^*_{LP}$

$w(I) \leq 4 \cdot \ln(n) \cdot Z^*_{LP} \leq 4 \cdot \ln(n) \cdot OPT$,
and the result is a set cover
The expected running time is polynomial.
(Calculations below)

p_i : prob. that e_i is covered

$\overline{p_i} = 1 - p_i$: prob. that e_i is not covered

Alg RR_1:



$$\overline{p_i} = \prod_{j: e_i \in S_j} \underbrace{(1-x_j)}_{\leq e^{-x_j}, \text{ for any } x_j \in \mathbb{R}}$$

$$\leq \prod_{j: e_i \in S_j} e^{-x_j} = e^{\overbrace{-\sum_{j: e_i \in S_j} x_j}^{\leq -1,}} \leq e^{-1} \approx 0.37$$

by the LP constraint corresponding to e_i

Alg RR_2:

$$\overline{p_i} \leq (e^{-1})^{2 \ln n} = e^{-2 \ln n} = (e^{\ln n})^{-2} = n^{-2}$$

$\Downarrow$

$$\Pr[\text{not set cover}] \leq \sum_{i=1}^{n} \overline{p_i} \leq \sum_{i=1}^{n} n^{-2} = n \cdot n^{-2} = n^{-1}$$

$$\underbrace{\Pr[w(I) \geq 4 \cdot \ln(n) \cdot z^*_{LP}]}_{> \frac{1}{2} \text{ would give } E[w(I)] > 2 \cdot \ln(n) \cdot z^*_{LP} \ \lightning} \leq \frac{1}{2}, \text{ by Markov's Inequality:}$$

Alg RR_3:

$$\Pr[\text{„not set cover" or „too expensive"}] \leq n^{-1} + \frac{1}{2}$$

Thus,

$$E[\#\text{iterations}] \leq \frac{1}{1 - (n^{-1} + \frac{1}{2})} \approx 2$$

Sometimes randomized algorithms are simpler / easier to describe/come up with.

Sometimes randomized algorithms can be derandomized as we saw in Chapter 5.

Exercise sheet 7: derandomize $\mathrm{Alg}\,RR_3$ (Ex. 5.7)

Exercise 5.7:

Derandomize the rounding alg. from Section 1.7, using the method of conditional expectations.

Hint: Use the following obj. fct. with random variables X_j, $1 \le j \le m$, and Z.

$$C = \sum_{j=1}^{m} X_j \cdot w_j + \lambda Z$$

$$\lambda = n \cdot \ln n \cdot Z^*_{LP}$$

$$X_j = \begin{cases} 1, & \text{if } S_j \text{ incl.} \\ 0, & \text{otherwise} \end{cases} \qquad Z = \begin{cases} 0, & \text{if set cover} \\ 1, & \text{otherwise} \end{cases}$$

With this obj. fct.,

any infeasible sol. has $C \geq \lambda = n \cdot \ln n \cdot Z^*_{LP}$ (*)

For $AlgRR_2$,

$$E[C] = E\Big[\sum_{j=1}^{m} X_j w_j\Big] + \lambda E[Z], \text{ by lin. of exp.}$$

$$\le 2 \cdot \ln n \cdot Z^*_{LP} + \cancel{n} \cdot \ln n \cdot Z^*_{LP} \cdot \cancel{n^{-1}}, \text{ by the analysis in Sec. 1.7}$$

$$= 3 \cdot \ln n \cdot Z^*_{LP}$$

Thus, using the method of cond. exp., we can find a sol with $C \le E[C] \le 3 \cdot \ln n \cdot Z^*_{LP}$, and by (*), such a sol. is a set cover (assuming $n > 3$).

In order to do this, we must be able to calculate conditional exp values, i.e., calculate $E[C]$, given that decisions about $S_1, \dots, S_\ell$ have already been made:

Let $\vec{X}_\ell = (X_1, X_2, \dots, X_\ell)$. Then,

$$E[C \mid \vec{X}_\ell] = \sum_{j=1}^{\ell} X_j w_j + \sum_{j=\ell+1}^{m} x_j w_j + \lambda E[Z \mid \vec{X}_\ell]$$

where $E[Z \mid \vec{X}_\ell]$ can be calculated in the following way.

For each element e_i,

$$Pr[e_i \text{ covered} \mid \vec{X}_\ell] = \begin{cases} 1, & \text{if } e_i \text{ is contained in a set } S_j \text{ s.t. } j \le \ell \text{ and } X_j = 1 \text{ (i.e., } e_i \text{ is covered by one of the sets } S_1, \dots, S_\ell) \\ \underbrace{1 - \prod_{\substack{j: e_i \in S_j \\ \wedge j > \ell}} (1 - x_j)}_{\text{prob. that } e_i \text{ will } \underline{\text{not}} \text{ be covered by any of the sets } S_{\ell+1}, \dots, S_m}, & \text{otherwise} \end{cases}$$

$$\begin{aligned} E[Z \mid \vec{X}_\ell] &= Pr(\text{set cover}) \cdot 0 + Pr(\text{not set cover}) \cdot 1 \\ &= Pr(\text{not set cover}) \\ &= 1 - \underbrace{\prod_{i=1}^{n} Pr[e_i \text{ covered} \mid \vec{X}_\ell]}_{Pr(\text{set cover})} \end{aligned}$$

$DeRR_2$

Solve LP optimally

For $l \leftarrow 1$ to m

  If $E[C \mid (X_1, X_2, \ldots, X_{l-1}, 0)] \leq E[C \mid (X_1, X_2, \ldots, X_{l-1}, 1)]$

    $X_l \leftarrow 0$

  Else

    $X_l \leftarrow 1$

Sheet 7:

1. Primal-dual for unweighted VC

Primal:

$$\min \sum_{v \in V} x_v$$

$$\text{s.t.} \quad x_u + x_v \geq 1, \qquad (u,v) \in E$$

$$0 \leq x_v \leq 1, \qquad v \in V$$

Dual:

$$\max \sum_{e \in E} y_e$$

$$\text{s.t.} \quad \sum_{e \in Adj(v)} y_e \leq 1, \qquad v \in V$$

$$0 \leq y_e \leq 1, \qquad e \in E$$

a) What does the alg. do?

For each $e \in E$: $y_e \leftarrow 0$
While some edge (u,v) is not covered
  $y_{(u,v)} \leftarrow 1$ // The two dual constr. corr. to u and v become tight
  Select u and v

b) Alg. without mention of LP

While some edge (u,v) is not covered
  Select both endpoints u and v

c) Lower bound on approx. factor